Problem Sheet 9

Problem 1

Every p-adic number $0 \neq \alpha \in \mathbb{Q}_p$ has a unique series expansion

$$\alpha = \sum_{i=n}^{\infty} a_i p^i, \quad n \in \mathbb{Z},\ a_i \in \{0, \ldots, p-1\},\ a_n \neq 0.$$

(a) Prove that $\alpha \in \mathbb{Q}$ if and only if the sequence $(a_i)_{i \geq n}$ is eventually periodic.

(b) Determine the 5-adic expansion of $2/35$.

Problem 2

The following is a variant of Hensel's Lemma.

(a) Let $A \twoheadrightarrow B$ be a surjection of rings with nilpotent kernel. Let $f \in A[T]$ with derivative f' and let $\alpha \in B$ be such that

$$f(\alpha) = 0, \quad f'(\alpha) \in B^{\times}.$$

Prove that there is a unique lift $\tilde{\alpha} \in A$ of α with $f(\tilde{\alpha}) = 0$.

(b) Show that $\sqrt{2}$ exists in $\mathbb{Q}_7$. Show similarly that $T^3 - 2 = 0$ has a unique solution in $\mathbb{Q}_5$.